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Temperature Overshoots for a 4-Velocity Unidimensional Discrete Boltzmann Model

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We propose a 4-velocity unidimensional discrete Boltzmann model with two different speeds 2, 1 and two different masses 1, 2. With the three conservation laws of mass, momentum, and energy satisfied, we can introduce a nontrivial temperature. First, we determine the similarity shock waves satisfying physical properties: positivity, shock stability, inequalities of the subsonic and supersonic flows, increase or decrease of both mass and temperature across the shock. It results that either the speed of the shock front is higher than the speed 1 of the slow particles and the shocks are compressive or less than 1 and the shocks are rarefactive. We observe overshoots of the temperature, across the shock, with bumps higher and higher as the shock front speed increases. Second, we study the $(1+1)$-dimensional shock waves. They represent the superposition and collision of two compressive shocks traveling in opposite directions and we observe temperature overshoots for not too large times.

KEY WORDS: Discrete Boltzmann models; shockwave solutions.

1. INTRODUCTION

For the discrete-velocity Boltzmann models[1] along an axis $0\mathbf{x}$, the velocity $\mathbf{V}$ takes only a finite number of discrete values $\mathbf{V}_i$: $i=1,\ldots,p$. To each velocity $\mathbf{V}_i$ is associated a density N_i satisfying a nonlinear equation, so a system of p nonlinear equations for a model with p velocities. In order to be physically relevant, these models must satisfy the three linear conservation laws of mass $\mathcal{M}(x,t)$, momentum $\mathcal{J}(x,t)$, and energy $\mathcal{E}(x,t)$. These

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macroscopic quantities are linear combinations of the microscopic densities $N_i(x, t)$. So the corresponding linear combination of the nonlinear N_i equations must reproduce the three linear conservation laws. Consequently, a nonlinear unidimensional Boltzmann model must contain at least four discrete velocities ($p \geqslant 4$).

In fact, the current unidimensional models[(2)] violate the momentum conservation law. In general they are two-velocity discrete models with $\mathbf{V}_1 + \mathbf{V}_2 = 0$ and only one speed $|\mathbf{V}_1| = |\mathbf{V}_2| = 1$. For these models, since the mass $\mathscr{M} = \sum N_i$ and the energy $\mathscr{E} = \sum N_i \mathbf{V}_i^2/2$ are proportional, the temperature $\mathscr{T}e(x, t) = 2\mathscr{E}/\mathscr{M} - \mathscr{U}^2$, $\mathscr{U} = \mathscr{J}/\mathscr{M}$, cannot be distinguished from the velocity $\mathscr{U}(x, t)$. For the temperature this is a general drawback of all the discrete models (unidimensional, planar, three-dimensional) with only *one speed* $|V_i| = 1$.

For the two-velocity discrete models, the known[3] shock wave solutions are the similarity waves, but no $(1+1)$-dimensional solutions are known (except for the completely soluble Ruijgrook–Wu model). For these models, the temperature cannot be defined. Here our goal is to introduce and study the temperature.

We propose a 4-velocity unidimensional model with *two speeds* and two different masses for the particles. Both the H-theorem and the three independent conservation laws are satisfied (which allows one to define a temperature). For this model, with two couples of opposite velocities, $\mathbf{V}_1 + \mathbf{V}_2 = \mathbf{V}_3 + \mathbf{V}_4 = 0$, $\mathbf{V}_1$ and $\mathbf{V}_3$ along the positive x axis, we have two speeds $|\mathbf{V}_i|$ and two different masses for the particles: $|\mathbf{V}_i| = 2$, $m_i = 1$ for $i = 1, 2$ and $|\mathbf{V}_i| = 1$, $m_i = 2$ for $i = 3, 4$. The associated microscopic densities satisfy a system of nonlinear equations which include, by linear combination, the three linear conservation laws for the macroscopic quantities $\mathscr{M}$, $\mathscr{J}$, $\mathscr{U}$, $\mathscr{E}$, $\mathscr{T}e$,

$$
\begin{gathered}
l_1 N_1 = l_4 N_4 = -l_2 N_2 = -l_3 N_3 = N_2 N_3 - N_1 N_4 \\
l_i = \partial_t + a_i \partial_x, \qquad a_1 = -a_2 = 2, \qquad a_3 = -a_4 = 1 \\
\mathscr{M} = N_1 + N_2 + 2N_3 + 2N_4 \\
\mathscr{J} = 2(N_1 - N_2 + N_3 - N_4) \\
\mathscr{E} = 2N_1 + 2N_2 + N_3 + N_4 \\
\mathscr{U} = \mathscr{J}/\mathscr{M} \\
\mathscr{T}e = 2\mathscr{E}/\mathscr{M} - \mathscr{U}^2 \\
\mathscr{M}_t + \mathscr{J}_x = 0 \qquad \mathscr{J}_t + 2\mathscr{E}_x = 0, \qquad \mathscr{E}_t + \mathscr{Q}_x = 0, \qquad \mathscr{Q} = 4N_1 - 4N_2 + N_3 - N_4
\end{gathered}
\tag{1.1}
$$

[3] See ref. 3 for the two-velocity models; see ref. 4 for a recent review.

We notice that a lattice gas model[5] similar to the present discrete model (1.1) has been recently studied. A discrete model of the (1.1) type for a gas mixture with two different species and two different sets of macroscopic quantities was also previously studied[6] for the possible existence of entropy overshoots[7] in shocks. Here, in (1.1), we study the macroscopic quantities of the whole gas, a sum of the two different species; a preliminary note was presented.[8] Finally, we notice also that temperature overshoots have been observed for two discrete models with temperature: $8\mathbf{V}_i$ and two speeds,[8] and $9\mathbf{V}_i$ and three speeds, which were previously studied as lattice gas models.[9]

In Section 2 we study the exponential-type similarity shock waves:

$$N_i = n_{0i} + n_i/D, \qquad D = 1 + e^{\gamma\eta}, \qquad \eta = x - \xi t \tag{1.2}$$

Since the pioneering work of Broadwell,[10] it has been recognized that the most interesting physical application of the discrete kinetic theory is the study of the shock waves. Here, in addition to the previous studies,[1,3,5,10,11] we introduce the temperature. The first physical constraint for the physical relevance of the shock wave solutions is the *positivity of the densities*. Consequently, we prove that the speed $|\xi|$ of the shock front cannot exceed the value 2 of the speed of the fast particles. For the physically acceptable classes of positive solutions we show that distinctions occur between $|\xi|$ less or higher than the speed 1 of the slow particles and between $\xi > 0$ and $\xi < 0$. If we define the two Maxwellian states Ma_0, Ma_s by the two sets n_{0i} and $s_{0i} = n_{0i} + n_i$, we find that the four possible ξ intervals are limited by the velocities ± 1, ± 2 of the slow and fast particles and by the parameters of the two Maxwellian states. The second physical constraint concerns the *stability of the upstream and downstream Maxwellian states* and the determination of the direction of the shock. We introduce the characteristic values ξ_0, ξ_s for a weak shock, and determine both the shock velocities $\mathcal{U}_0 - \xi$ and $\mathcal{U}_s - \xi$ ($\mathcal{U}_0$ and $\mathcal{U}_s$ are the velocities of the Maxwellian states) and the sound speeds $\mathcal{U}_0 - \xi_0$ and $\mathcal{U}_s - \xi_s$. We prove that the inequalities for subsonic and supersonic flows are satisfied. The result is that for $|\xi|$ higher than the slow particles of speed 1, only compressive shock can occur (mass increasing across the shock), while for $|\xi| < 1$ we find both compressive and rarefactive shocks (mass decreasing across the shock). As a third physical requirement we *introduce the temperature* and ask that both the mass and the temperature either increase together (compressive shock) or decrease together (rarefactive shock). As a consequence only the compressive shocks remain for $|\xi| > 1$ and rarefactive shocks for $2/\sqrt{5} < |\xi| < 1$. All these results, analytically proved, require a lot of tedious calculations, which are provided in Appendix A. Finally, we study the *possibility of temperature overshoots* in the interior of the shock waves. The mass, momentum, etc., are monotonic η functions between the two Maxwellian states; on the contrary, the temperature, which is the difference between the two monontonic η-functions $2\xi/\mathscr{M}$ and $\mathscr{U}^2$, is not necessarily monotonic. A simple criterion for such an effect is that the temperature inside the shock front at $\eta=0$ is higher than its values at the two Maxwellian states. We observe such overshoots and see that they become more pronounced when the shock front speed $|\xi|$ increases. We also observe overshoots of the local entropy.[7]

In Section 3 and Appendix B we study the $(1+1)$-dimensional shock waves which are sums of two similarity shock waves:

$$N_i = n_{0i} + \sum_{j=1}^{2} n_{ji}/D_j, \qquad D_j = 1 + d_j e^{\gamma_j \eta_j}, \quad \eta_j = x - \xi_j t \tag{1.3}$$

We find that the shock front velocities ξ_j of the two components are opposite and that their speed $|\xi_j|$ is higher than the speed 1 of the slow particles. The physical interpretation is that the $(1+1)$-dimensional solutions represent the superposition and the collision of two compressive shocks traveling in opposite directions. We still see temperature overshoots for not too large times. Contrary to the similarity shock waves, the shock profiles are modified when the time is growing and a relaxation toward a third Maxwellian equilibrium state is observed in addition to the two Maxwellian shock states. These features are the same for both the mass and the temperature.

2. SIMILARITY SHOCK WAVES

In Appendix A we first recall the known result[3] for the possible exponential-type similarity waves, leading to similarity solutions

$$N_i = n_{0i} + n_i/D, \qquad D = 1 + de^{\gamma\eta}, \quad \eta = x - \xi t \tag{2.1}$$

of the nonlinear equations (1.1) satisfied by our $4\mathbf{V}_i$ model. We build up the macroscopic quantities associated with the densities (2.1): mass $\mathscr{M} = M_0 + M/D$, momentum $\mathscr{J} = J_0 + J/D$, energy $\mathscr{E} = E_0 + E/D$, velocity $\mathscr{U} = \mathscr{J}/\mathscr{M}$, shock velocity $\mathscr{V} = \mathscr{U} - \xi$, temperature $\mathscr{T}e = 2\mathscr{E}/\mathscr{M} - \mathscr{U}^2$, and sound velocity $\mathscr{W}$. All details are provided in Appendix A and we briefly report the main results.

First, we construct four classes of positive densities. They are characterized by the speed $|\xi|$ of the shock front being less or higher than 1 and by the sign of ξ. We prove that $|\xi|$ cannot exceed the speed 2 of the fast particles. ξ belongs to four intervals with limits given by ± 1 and ± 2 and by the parameters of the two Maxwellian states Ma_0 and Ma_s defined, respectively, by n_{0i} and $s_{0i} = n_{0i} + n_i$.

Second, we introduce the characteristic ξ_0, ξ_s values for the weak shocks associated with Ma_0 and Ma_s and show that both ξ, ξ_0 and ξ, ξ_s belong to the above four intervals. However, only three characteristic values belong to three of these four intervals.

Third, we study the velocities U_0, U_s, the shock velocities $V_0 = U_0 - \xi$, $V_s = U_s - \xi$, and the sound wave velocities $W_0 = U_0 - \xi_0$, $W_s = U_s - \xi_s$ associated with the Maxwellians. From the signs of both the shock velocity and γ we find when $|\eta| \to \infty$ which Maxwellian is in the upstream or downstream domains. For the classes $1 < |\xi| < 2$, with shock front speed higher than the speed of the slow particles, we find that only compressive shock (mass increasing across the shock) can occur. On the contrary, for $0 < |\xi| < 1$ with shock front speed less than the fast particles both compressive and rarefactive (mass decreasing across the shock) shocks exist.

Fourth, another distinction between the possible classes of shocks arises, depending upon whether the mass M_0 of the Ma_0 is higher or less than the mass $M_s = M_0 + M$ of the other Ma_s. In principle, for this simple model we could have 12 subclasses of solutions. Fortunately, invariance properties allow us to study only three subclasses. For the stability of the Maxwellian states we verify that the subsonic and supersonic flow inequalities are satisfied.

Fifth we introduce the temperature and require that for compressive or rarefactive shocks, both mass and temperature increase or decrease together between the two Maxwellian states. We study the possibility of an overshoot of the temperature across the shock and we define a criterion for this effect.

Finally, as an illustration we present some numerical calculations.

2.1. Algebraic Construction of the Solutions

The ten parameters n_{0i}, n_i, γ, ξ satisfy six relations, leaving four arbitrary parameters

$$\xi, \quad n_{0i} > 0, \quad i = 1, 2, 3 \tag{2.2}$$

Always one parameter is a scaling one, so that we could, for instance, put $n_{01} = 1$. For the construction of the nonarbitrary parameters it is convenient to introduce ξ-dependent intermediate parameters $\bar{n}_i = n_i/n_1$, $\bar{\gamma} = \gamma/n_1$,

$$\bar{n}_2 = (2-\xi)/(2+\xi), \qquad \bar{n}_3 = (2-\xi)/(\xi-1), \qquad \bar{n}_4 = (\xi-2)/(\xi+1)$$

$$\bar{\gamma} = 2\xi/(2+\xi)(\xi^2-1) \tag{2.3}$$

while n_1 depends on the four arbitrary parameters

$$n_1(2-\xi)\,\bar{\gamma}=n_{04}+n_{01}\bar{n}_4-n_{02}\bar{n}_3-n_{03}\bar{n}_2, \qquad n_{04}=n_{03}n_{02}/n_{01}>0 \tag{2.4}$$

With (2.3)–(2.4) we construct the original $n_i=\bar{n}_i n_1$, $\gamma=\bar{\gamma}n_1$. Finally, the n_{0i} satisfy the Maxwellian relation for Ma_0, while for Ma_s we deduce $s_{04}s_{01}=s_{02}s_{03}$ with $s_{0i}=n_{0i}+n_i$.

2.2. Invariance Properties under the Transforms $\mathscr{T}_1$ and $\mathscr{T}_2$

The relations satisfied by the parameters are invariant under the transforms

$$\mathscr{T}_1:\quad \xi\to-\xi,\quad \gamma\to-\gamma,\quad n_i\leftrightarrow n_{i+1},\quad n_{0i}\leftrightarrow n_{0i+1},\quad i=1 \text{ and } 3 \tag{2.5}$$

For instance, we can study the solutions with $\xi>0$ and deduce the $\xi<0$ ones. Under this transform we find for the microscopic densities $N_i(x,t)\leftrightarrow N_{i+1}(-x,t)$, $i=1$ and 3, while for the macroscopic $\mathscr{M}$, $\mathscr{J}$, $\mathscr{E}$, $\mathscr{T}e$, $\mathscr{U}$, which are functions of x, t, we obtain $\mathscr{M}(-x,t)$, $-\mathscr{J}(-x,t)$, $\mathscr{E}(-x,t)$, $\mathscr{T}e(-x,t)$, $-\mathscr{U}(-x,t)$. A second transform $\mathscr{T}_2$ interchanges the Maxwellians Ma_0 and Ma_s,

$$\mathscr{T}_2:\quad n_{0i}\to s_{0i},\quad n_i\to s_i=-n_i,\quad \gamma\to-\gamma,\quad \xi\to\xi \tag{2.6}$$

We obtain $\mathscr{T}_2 N_i=N_i$ and $\mathscr{T}_2 M_0=M_s=M_0+M$. For a solution with ξ fixed but $M_0\lesseqgtr M_1$ we can obtain the other one with $M_0\gtreqless M_s$.

2.3. Classes of Positive Densities $N_i>0$ or $n_{0i}>0$, $s_{0i}>0$

From (2.2), all n_{0i} are positive. For the s_{0i} [see (A.7)] we always obtain the same analytic structure

$$n_{0i}s_{0i}=\Gamma_i(\xi)\,[n_{02}-n_{01}\alpha_i(\xi)][n_{03}-n_{01}\beta_i(\xi)]$$

so that we check the signs of Γ_i, α_i, β_i. We find for n_{02}/n_{01}, n_{03}/n_{01} lower and upper ξ-dependent bounds leading to $s_{0i}>0$ and $N_i>0$. Positivity is violated for $|\xi|>2$, which means that the shock front cannot travel faster than the fast particles with speed 2. For positive $\xi<2$ we find two classes of positive densities (A.8a),

$$\text{Class I}\qquad 1<\xi<2,\quad n_{02}>\bar{n}_2 n_{01},\quad n_{03}>\bar{n}_3 n_{01} \tag{2.7a}$$

$$\text{Class III}\qquad 0<\xi<1,\quad \bar{n}_4/\bar{n}_3<n_{02}/n_{01}<\bar{n}_2 \tag{2.8a}$$

while applying the transform $\mathscr{T}_1$, we deduce the two other classes:

Class II $\quad -2<\xi<-1, \quad n_{02}<\bar{n}_2 n_{01}, \quad n_{03}>n_{01}\bar{n}_4/\bar{n}_2$

Class IV $\quad -1<\xi<0, \quad \bar{n}_2<n_{02}/n_{01}<\bar{n}_4/\bar{n}_3$

Equivalently we can write the positivity constraints on the front shock velocity ξ in terms of the parameters n_{0i}, s_{0i} of the two Maxwellians Ma_0 and Ma_s [see (A.8b)–(A.8c)]. We define $a(n_{0i})=1+n_{01}/(n_{01}+n_{03})>1$, $b(n_{0i})=(n_{01}-n_{02})/(n_{01}+n_{02})$, and $a(s_{0i})>1$, $b(s_{0i})$ with s_{0i} instead of n_{0i}:

Class I $\quad \sup(a(n_{0i}), 2b(n_{0i}))<\xi<2 \qquad (2.7b)$

$\qquad \sup(a(s_{0i}), 2b(s_{0i}))<\xi<2 \qquad (2.7c)$

Class III $\quad n_{02}<n_{01}, \quad 0<b(n_{0i})<\xi<\inf(1, 2b(n_{0i})) \qquad (2.8b)$

$\qquad s_{02}<s_{01}, \quad 0<b(s_{0i})<\xi<\inf(1, 2b(s_{0i})) \qquad (2.8c)$

The two other classes II and IV for $\xi<0$ are respectively obtained by application of the transform $\mathscr{T}_1$ to the classes I and III. We notice that for a Maxwellian given (either the set n_{0i} or the set s_{0i}), only three intervals for ξ are possible: one for class I, another for class II, and the third one either for class III or IV, depending upon whether $n_{02}\lesseqgtr n_{01}$, $s_{02}\lesseqgtr s_{02}$.

2.4. Characteristic Velocities for Weak Shocks

Let us call ξ_0 and ξ_s the ξ values for weak shocks associated, respectively, with Ma_0 and Ma_s. We begin with ξ_0, for which $n_i(\xi_0)=0$ for all i values, and define $\hat{n}_1(\xi, n_{0i})$:

$$\begin{aligned} 2\xi(\xi-2)\, n_1=\hat{n}_1&=(1-\xi^2)(n_{04}+n_{01}\bar{n}_4-n_{02}\bar{n}_3-n_{03}\bar{n}_2)(2+\xi) \\ \hat{n}_1&=(1-\xi^2)[n_{04}(2+\xi)-n_{03}(2-\xi)] \\ &\quad+(4-\xi^2)[n_{02}(1+\xi)-n_{01}(1-\xi)] \end{aligned} \tag{2.9}$$

$\hat{n}_1(\xi_0)=0$ are the three roots of a cubic polynomial. What is important and proved in Appendix A is that ξ and ξ_0 belong to the same interval, either the one of class I defined in (2.7b) or the corresponding one of class II [see (A.8b)], and, finally, the one defined in (2.8b) for class III if $n_{02}<n_{01}$ or the corresponding one of class IV if $n_{01}<n_{02}$. We go on with the Maxwellian Ma_s; define $\hat{s}_1(\xi, s_{0i})$

$$\hat{s}_1(\xi)=(2+\xi)(1-\xi^2)(s_{04}+s_{01}\bar{n}_4-s_{02}\bar{n}_3-s_{03}\bar{n}_2)=-\hat{n}_1(\xi) \tag{2.10}$$

and ξ_s are the three roots $\hat{s}_1(\xi_s)=0$ of the cubic $\hat{s}_1(\xi)$ polynomial. Here also ξ, ξ_s belong to three similar intervals: (2.7c) and the corresponding one for class II and either (2.8c) for class III if $s_{02} < s_{01}$ or the corresponding one for class IV if $s_{01} < s_{02}$.

2.5. Velocity, Shock Velocity, and Sound Velocity

To the mass $\mathcal{M} = M_0 + M/D$, momentum $\mathcal{J} = J_0 + J/D$, velocity $\mathcal{U} = \mathcal{J}/\mathcal{M}$, and shock velocity $\mathcal{V} = \mathcal{U} - \xi$ we associate ($D \to \infty$ or 1) the corresponding quantities for the two Maxwellian states:

$$Ma_0\colon \quad M_0,\, J_0,\, U_0 = J_0/M_0,\, V_0 = U_0 - \xi$$

$$Ma_s\colon \quad M_s = M_0 + M,\, J_s = J_0 + J,\, U_0 = M_s/J_s,\, V_s = U_s - \xi$$

They are linked by the mass conservation law

$$M_s V_s = M_0 V_0 \to V_0 V_s > 0 \qquad \text{and} \qquad |V_0| \gtrless |V_s| \quad \text{if} \quad M_0 \lessgtr M_s \tag{2.11}$$

and V_0, V_s have the same sign. Depending upon whether $\gamma\eta \to +\infty$ or $-\infty$, the Maxwellian states are either Ma_0 or Ma_s, so that the γ sign gives the information for the $|\eta| \to \infty$ states. Further, the V_0 (or V_s) sign gives the direction of the shock. With this knowledge we can define the upstream and downstream states

$$V_0 M_0 = 2(n_{03}/n_{01})(\xi + 1)(n_{01}\bar{n}_4/\bar{n}_3 - n_{02}) + (2 + \xi)(\bar{n}_2 n_{01} - n_{02}), \qquad M\xi = 6\gamma \tag{2.12}$$

First, for class I with $\xi > 1$ and $n_{02} > \bar{n}_2 > \bar{n}_4/\bar{n}_3$ we find both $M\gamma > 0$ and $V_0 < 0$. Depending upon whether $M_s - M_0 = M \gtrless 0$, we have $\gamma \gtrless 0$, and Ma_0, Ma_s are the shock limits when $\eta \to \pm\infty$. We obtain two subclasses:

$$\begin{aligned} &\text{Class IA} \quad M > 0, \quad \gamma > 0, \quad -V_0 > -V_s > 0; \quad \text{up } \eta = \infty\ Ma_0, \\ &\qquad\qquad \text{down } \eta = -\infty\ Ma_s \\ &\text{Class IB} \quad M < 0, \quad \gamma < 0, \quad -V_s > -V_0 > 0; \quad \text{up } \eta = \infty\ Ma_s, \\ &\qquad\qquad \text{down } \eta = -\infty\ Ma_0 \end{aligned} \tag{2.7d}$$

and notice that class IB can be obtained from class IA by the transform $\mathcal{T}_2$ which interchanges the two Maxwellian states. In both subclasses the shock is compressive because the mass increases across the shock (mass downstream larger than mass upstream).

Second, for class II with $-2 < \xi < -1$ we have $M\gamma < 0$ and, applying the transform $\mathcal{T}_1$, we find $V_0 > 0$, $V_s > 0$. We still have two subclasses A and B corresponding to $M \gtrless 0$, and applying $\mathcal{T}_1$ to (2.7d), we verify that the shock is still compressive.

Third, for class III with $0<\xi<1$ we still have $M\xi>0$ with two subclasses $M \gtrless 0$, but the discussion about the V_0 sign is more complicated. V_0 does not have a well-defined sign in (2.12), and defining $\tilde{n}_3$, we find

$$\text{Class III} \qquad \tilde{n}_3 = [(2-\xi)\,n_{01} - (2+\xi)\,n_{02}]/2[(1+\xi)\,n_{02} - (1-\xi)\,n_{01}]$$
$$\rightarrow V_0 \gtrless 0 \qquad \text{if} \quad n_{03} \lessgtr \tilde{n}_3 n_{01}$$

For M fixed with $M_0 \gtrless M_s$ and two possible directions of the shock, necessarily one shock is compressive while the other is rarefactive (mass decreasing across the shock). In Section A.8 we present the two subclasses IIIA, $V_0 \gtrless 0$, and IIIB, $V_0 \gtrless 0$, where we have

$$\text{Class IIIB} \qquad M<0, \quad \gamma<0, \quad \eta=\infty\ Ma_s, \quad \eta=-\infty\ Ma_0,$$
$$M_s<M_0; \quad V_0<0$$

if $n_{03}>\tilde{n}_3 n_{01}$, up Ma_0, down Ma_s, compressive shock; $V_0>0$ if $n_{03}<\tilde{n}_3 n_{01}$, up Ma_s, down Ma_0 rarefactive shock.

The four subclasses IV, $M \gtrless 0$, $V_0 \gtrless 0$, with rarefactive and compressive shocks are obtained by applying $\mathscr{T}_1$ to the subclasses III. At this stage of our study there exists a great difference between the shocks propagating with speeds greater or less than the slow particle speed. In the first case, only compressive shocks can occur, while in the second case, both compressive and rarefactive shocks are possible.

Among the 12 subclasses, applying the transforms $\mathscr{T}_1$ and $\mathscr{T}_2$, only three of them (one of class I and two of class III) generate all the others. For the stability of the solutions in the upstream and downstream domains (Lax–Whitham stability theory,[11] Gatignol[1–11]) it is sufficient to check the inequalities of the subsonic and supersonic flows.

We define the sound velocity $W_0=U_0-\xi_0$, $W_s=U_s=U_s-\xi_s$ associated to the Maxwellians Ma_0, Ma_s and compare with the shock velocity $V_0=U_0-\xi$, $V_s=U_s-\xi$. For a supersonic flow we must have $|V_0|>|W_0|$ or $|V_s|>|W_s|$, while for a subsonic flow $|V_0|<|W_0|$ or $|V_s|<|W_s|$. In Lemmas 1–3 of Appendix A, for the three generating subclasses class I1, class IIIB, $V_0 \gtrless 0$, we prove that the supersonic and subsonic inequalities are satisfied.

2.6. Energy and Temperature

For these models with two speeds, mass and energy conservation laws are different, so that we can introduce nontrivial energy and temperature macroscopic quantities. New physical constraints will occur for our previous classes of shock solutions. For a compressive shock we will require that both mass and temperature increase across the shock (Their product, which is the pressure, will increase, too.) Similarly, we will require that they both decrease for a rarefactive shock and the pressure will decrease. We shall see that this physical condition cannot be satisfied for the compressive shocks of classes III and IV, for which $|\xi| < 1$.

We introduce the energy $\mathscr{E} = E_0 + E/D$ with $2EM = J^2$ [mass conservation law (A.15)] and the temperature $\mathscr{T}e = 2\mathscr{E}/\mathscr{M} - \mathscr{U}^2$:

$$\mathscr{T}e = (\mathscr{N}_0 + \mathscr{N}/D)/(M_0 + M/D)^2$$
$$\mathscr{N}_0 = 2E_0 M_0 - J_0^2 > 0, \qquad \mathscr{N} = 2MC, \qquad C = E_0 - \xi J_0 + \xi^2 M_0/2 > 0 \tag{2.13}$$

To the Maxwellians Ma_0 and Ma_s we associate the temperatures $\mathscr{T}^e{}_0$ and $\mathscr{T}e_s$:

$$\mathscr{T}e_0 = \mathscr{N}_0/M_0^2, \qquad \mathscr{T}e_s = (\mathscr{N}_0 + \mathscr{N})/M_s^2$$
$$\text{sign } \mathscr{T}e_0 - \mathscr{T}e_s = M[m + 2(1 - 1/\mu)], \qquad \mu = M_0 C/\mathscr{N}_0 > 0, \qquad m = M/M_0 \tag{2.14}$$

and m has the M sign. We require that $\mathscr{M}$ and $\mathscr{T}e$ increase or decrease together across the shock,

$$M_0 \gtrless M_s \rightarrow \mathscr{T}e_0 \gtrless \mathscr{T}e_s \qquad \text{or} \qquad m + 2(1 - 1/\mu) \gtrless 0 \quad \text{if} \quad M \gtrless 0$$

In Appendix A we check this property for the solutions of class III for which $0 < \xi < 1$. In Lemma 4 it is shown that this property is not possible for compressive shock. In Lemma 5 the same result holds for the rarefactive shocks if $\xi < 2/\sqrt{5} = 0.89$. An application of the transform $\mathscr{T}_1$ will give the same results for class IV. Consequently, in the following, we only consider compressive shock solutions of classes I and II with $1 < |\xi| < 2$ and rarefactive shocks of classes III and IV with $2/\sqrt{5} < |\xi| < 1$.

2.7. Overshoot of the Temperature

Let us neglect the velocity $\mathscr{U}$ in the temperature $\mathscr{T}e \simeq 2\mathscr{E}/\mathscr{M}$. Then it is shown in Appendix A that $\mathscr{T}e$ becomes a monotonic η-dependent function like the mass $\mathscr{M}(\eta)$ and the energy $\mathscr{E}(\eta)$. Adding $-\mathscr{U}^2$, we find that the whole temperature is not necessarily monotonic across the shock. We look at the possibility of an overshoot of $\mathscr{T}e$. A simple criterion for such an effect is

$$\mathscr{T}e(\eta = 0) = (\mathscr{N} + \mathscr{N}_0/2)/(M_0 + M/2)^2 > \sup\{\mathscr{T}e_0, \mathscr{T}e_s\}$$

Still assuming that mass and temperature increase or decrease together between the two Maxwellian states, we find that the criterion becomes

$$\text{Class A} \quad M > 0: \quad \mathcal{T}e_0 < \mathcal{T}e_s < \mathcal{T}e(0)$$

$$\text{Class B} \quad M < 0: \quad \mathcal{T}e_s < \mathcal{T}e_0 < \mathcal{T}e(0)$$

$$\operatorname{sign} \mathcal{T}e(0) - \mathcal{T}e_0 = M(-m/4 - 1 + 1/\mu)$$

$$\operatorname{sign} \mathcal{T}e(0) - \mathcal{T}e_s = M[3m/4 + 1 + (m^2/2 - 1)/\mu]$$

For instance, for class B we obtain the two conditions, with μ defined in (2.13)–(2.14),

$$\text{Class B} \quad M < 0: \quad 2 < (-M/M_0)\,\mu/(\mu - 1) < 4, \quad \mu > 1$$

which depend on ξ and on the macroscopic quantities M_0, J_0, E_0 of Ma_0. Recalling $M = M_s - M_a$, we see that the condition for the effect depends on the Maxwellian states and on the shock front speed. A similar condition for class A is written down in (A.25).

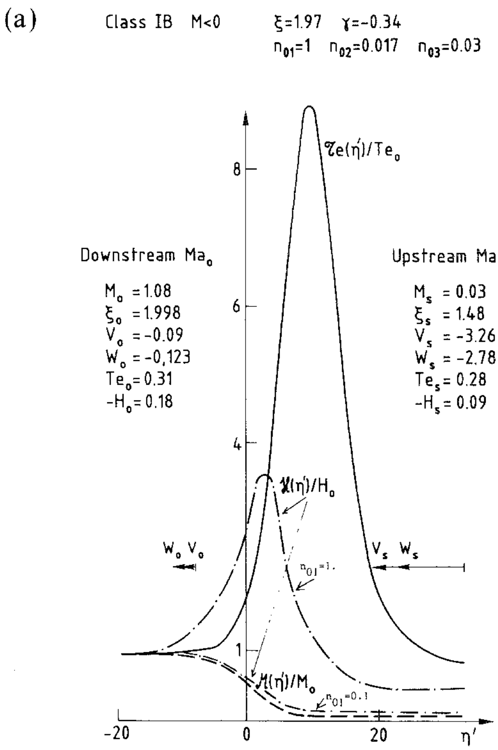
2.8. Entropies (Appendix A.9)

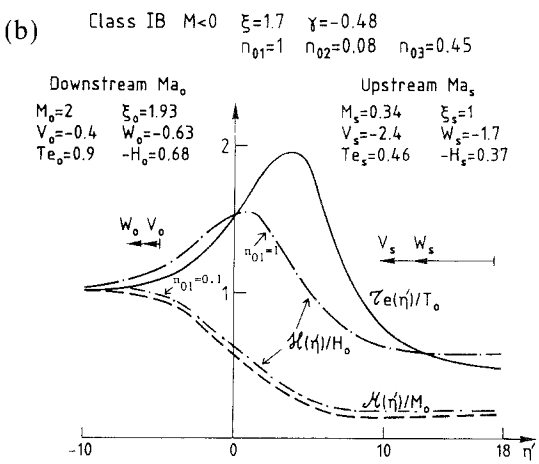
The shock functional $\mathcal{H}_1(\eta) = \sum(-\xi + a_i) N_i \log N_i$, $a_1 = -a_2 = 2$, $a_3 = -a_4 = 1$, is decreasing continuously between the two states $|\eta| = -\infty, +\infty$. On the contrary, the local entropy $-\mathcal{H}(\eta) = -\sum N_i \log N_i$ is not necessarily monotonic and can have overshoot or dip across the shock.

2.9. Numerical Calculations (Fig. 1)

We present, for two n_{01} values, the shock profiles both for compressive shocks with $|\xi| > 1$ and for one rarefactive shock with $\xi < 1$ (Fig. 1d). We quote $\mathcal{M}$, $\mathcal{T}e$, and $-\mathcal{H}$, as functions of $\eta' = \eta n_{01}$, normalized to their highest value $(M_0, Te_0, -H_0)$ either at the downstream state for compressive shocks or at the upstream state for the rarefactive shock. The supersonic and subsonic inequalities are satisfied and the values as $|\eta| \to \infty$ of the pressure $M_0 Te_0, M_s Te_s$ increase for compressive shocks and decrease for the rarefactive one. We require an increase of the local entropy between the two limits, or $-H_{\text{up}} < -H_{\text{down}}$. In Fig. 1 the quoted numbers are for $n_{01} = 1$, but for Ma_0 we have M_0/n_{01}, ξ_0, V_0, W_0, which are n_{01} independent, while H_0 is n_{01} dependent (the same for Ma_s). Consequently, $\mathcal{M}(\eta')/M_0$, $\mathcal{T}e(\eta')/Te_0$ are n_{01} independent, but $\mathcal{H}(\eta')/H_0$ is not.

For the compressive shocks we observe that the temperature overshoot increases with $|\xi|$ and we present the bumps for $\xi = 1.97$, 1.7, 1.5. On


(a)
Class IB M<0 ξ=1.97 γ=-0.34
n01=1 n02=0.017 n03=0.03
𝒯e(η′)/Te0
Downstream Ma0
M0 = 1.08
ξ0 = 1.998
V0 = -0.09
W0 = -0.123
Te0 = 0.31
-H0 = 0.18
Upstream Mas
Ms = 0.03
ξs = 1.48
Vs = -3.26
Ws = -2.78
Tes = 0.28
-Hs = 0.09
ℋ(η′)/H0
W0 V0
Vs Ws
n01=1.
ℳ(η′)/M0
n01=0.1
8
6
4
1
-20
0
20
η′


(b)
Class IB M<0 ξ=1.7 γ=-0.48
n01=1 n02=0.08 n03=0.45
Downstream Ma0
M0=2 ξ0=1.93
V0=-0.4 W0=-0.63
Te0=0.9 -H0=0.68
Upstream Mas
Ms=0.34 ξs=1
Vs=-2.4 Ws=-1.7
Tes=0.46 -Hs=0.37
W0 V0
Vs Ws
n01=1
n01=0.1
𝒯e(η′)/T0
ℋ(η′)/H0
ℳ(η′)/M0
2
1
-10
0
10
18
η′

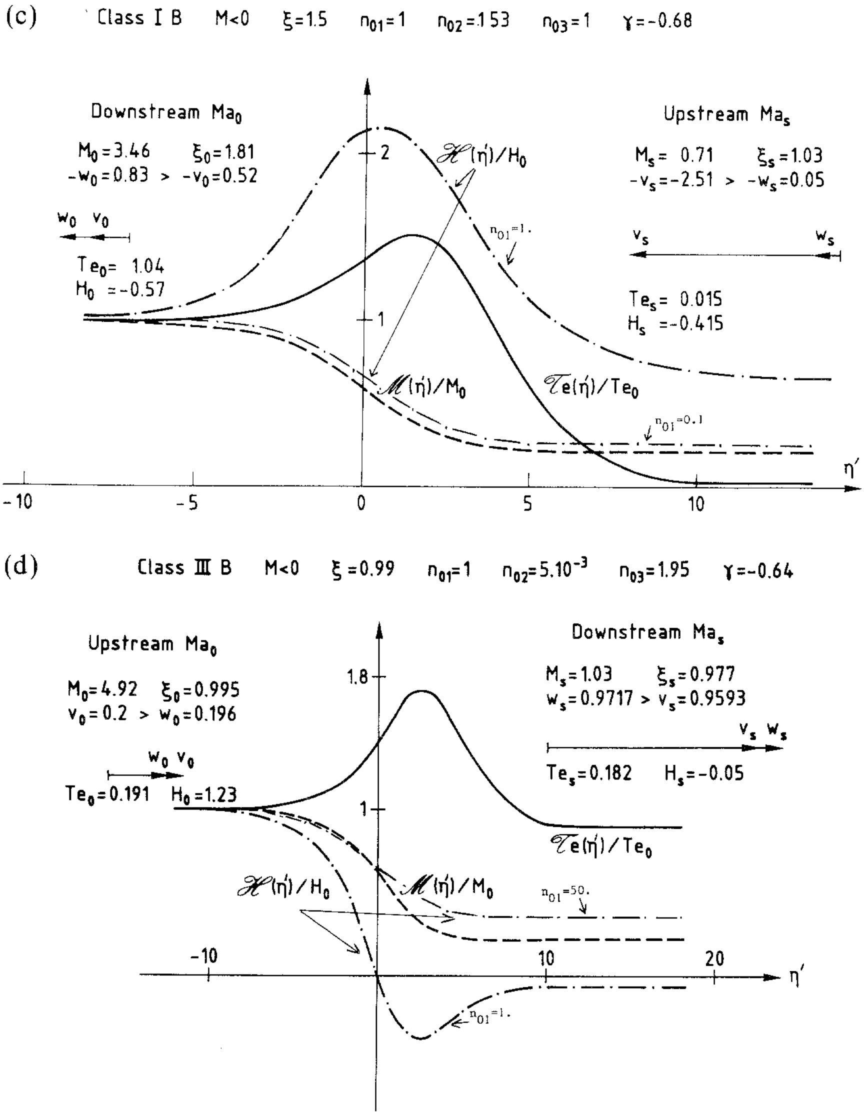


Fig. 1. Similarity shock wave functions of $\eta' = \eta n_{01}$. Here M_0/n_{01}, M_s/n_{01}, ξ_0, ξ_s, V_0, V_s, W_0, W_s, Te_0, Te_s, $\mathcal{M}(\eta')/M_0$, and $\mathcal{T}e(\eta')/Te_0$ are n_{01} independent, while H_0, H_s, $\mathcal{H}(\eta')/H_0$ are n_{01} dependent. The quoted n_{02}, n_{03}, M_0, H_0, M_s, and H_s values are for $n_{01} = 1$. (a–c) Compressive shocks for $\xi =$ (a) 1.97, (b) 1.7, (c) 1.5; and $n_{01} = 1$ and 0.1. Overshoots of $\mathcal{T}e(\eta')$ always exist, while overshoots of $\mathcal{H}(\eta')$ present for $n_{01} = 1$ disappear for smaller values, for instance, $n_{01} = 0.1$. For $n_{01} = 0.1$ and $\xi = 1.97$, 1.7, and 1.5 the H_0 values are, respectively, -0.26, -0.43, and -0.59 and the H_s values -0.014, -0.077, and -0.125. (d) Rarefractive shock, $\xi = 0.99$. A temperature overshoot exist, while for the local entropy a dip present at $n_{01} = 1$ disappears for large n_{01} values, for instance, $n_{01} = 50$, for whic $H_0 = 641$, $H_s = 200$.

the contrary, for $-\mathscr{H}$ the overshoot present for $n_{01}=1$ has disappeared for $n_{01}=0.1$. For the rarefactive shocks of class III–IV with $|\xi|<1$ we have numerically found that the joint decrease, between the Maxwellian states, of $\mathscr{M}$ and $\mathscr{T}e$ begins for $|\xi|\simeq 0.99$, showing that our theoretical evaluation result $|\xi|>2/\sqrt{5}$ is good enough. The temperature overshoot is less important than in the previous pictures with $|\xi|>1$, and in the example of Fig. 1d, $\xi=0.99$, while for $-\mathscr{H}$ the dip present for $n_{01}=1$ has disappeared for $n_{01}=80$.

In conclusion, for the compressive shocks, which are the standard physical shocks, the overshoot of the temperature effect becomes more and more important when the shock travels with its highest possible physical value.

3. (1+1)-DIMENSIONAL SHOCK WAVES

The exact (1 + 1)-dimensional solutions are the sums of two similarity waves[4–12]

$$N_i=n_{0i}+\sum_1^2 n_{ji}/D_j, \qquad D_j=1+d_j e^{\gamma_j \eta_j}, \qquad \eta_j=x-\xi_j t \tag{3.1}$$

If the two components are complex conjugate, the solutions are periodic[3,4,12,13] and such solutions exist for the present model.[8] Here we are interested in the temperature properties of shock waves and the two components are real. The two components $j=1$ and 2 must satisfy the similarity relations studied in Section 1. In addition, the sum must also be a solution and this gives another constraint [vanishing of the coefficient of $(D_1D_2)^{-1}$ in the collision terms]:

$$n_{12}n_{23}+n_{13}n_{22}=n_{11}n_{24}+n_{14}n_{21} \tag{3.2}$$

For the similarity solutions of Section 2 we have, in fact, only one variable $\eta=x-\xi t$ with two Maxwellian states when $|\eta|\to\infty$. On the contrary, for the (1 + 1)-dimensional solutions we really have two independent variables x and t, so that in addition to the two Maxwellian shock limits $|x|\to\infty$, the equilibrium Maxwellian state exists when $t\to\infty$. The associated physics is different. At initial time, or small time, we only observe the shock profile with two shock limits, but when the time is sufficiently large, the Maxwellian equilibrium state appears, which spreads out. The two similarity components $j=1, 2$ of (3.1) will be chosen as corresponding, respectively, to the two compressive shock classes I and II. The (1 + 1)-dimensional sum solution will represent the superposition or collision of two shock waves traveling in opposite direction with a relaxation toward equilibrium. All details are given in Appendix B.

3.1. Algebraic Construction of the Solutions

The 16 parameters $n_{0i}, n_i, \gamma_i, \xi_i$ satisfy the 11 similarity relations of the two components $j=2, 2$ plus another relation written down in (3.2). As in Section 2, we introduce ξ_j-dependent intermediate parameters $\bar{n}_{ji}=n_{ji}/n_{j1}, \bar{\gamma}_j=\gamma_j/n_{ji}$, deduce n_{j1}

$$\bar{n}_{j2}=(2-\xi_j)/(2+\xi_j), \qquad \bar{n}_{j3}=(2-\xi_j)/(\xi_j-1), \qquad \bar{n}_{j4}=(\xi_j-2)/(\xi_j+1)$$
$$(2+\xi_j)(\xi_j^2-1)\,\bar{\gamma}_j=2\xi_j, \qquad n_{j1}(2-\xi_j)\,\bar{\gamma}_j=n_{04}+n_{01}\bar{n}_{j4}-n_{02}\bar{n}_{j3}-n_{03}\bar{n}_{j2} \tag{3.3}$$

and reconstruct the other parameters $n_{ji}=\bar{n}_{ji}n_{j1}, \gamma_j=\bar{\gamma}_j n_{j1}$ from the five parameters $\gamma_j, n_{0i}, \ i=1, 2, 3$. However, we only have four arbitrary parameters, chosen to be

$$\xi_1, \quad n_{0i}>0, \quad i=1, 2, 3 \tag{3.4}$$

but (3.2) written with the intermediate parameters $\bar{n}_{12}\bar{n}_{23}+\bar{n}_{13}\bar{n}_{22}=\bar{n}_{14}+\bar{n}_{24}$ allows one to obtain ξ_2 from ξ_1. We obtain two classes of solutions [see (B.5)] and choose the simplest one,

$$\xi_1+\xi_2=0$$

Let us choose for the $j=1$ component the class I of Section 2 with $1<\xi_1<2$ and a negative shock velocity; then the $j=2$ component is of class II, $-2<\xi_2<-1$ with a positive shock velocity.

All Section 2 results concerning stability, sound wave velocity, and subsonic and supersonic inequalities are valid for the two components.

3.2. Positive (1+1)-Dimensional Solutions

If at initial time or at finite time, the asymptotic limits $|x|\to\infty$ are positive, then we can find[12] constraints on the d_j so that positivity holds for all x, t values. However, depending upon whether $\gamma_1\gamma_2 \lessgtr 0$, we find two sets of limits which must be positive:

$$\begin{aligned}&\gamma_1\gamma_2<0\to\Sigma_{ij}=n_{0i}+n_{ji}>0, \qquad i=1,\ldots, 4, \quad j=1, 2\\ &\gamma_1\gamma_2>0\to n_{0i}>0, \qquad \Omega_i=n_{0i}+n_{1i}+n_{2i}>0\end{aligned} \tag{3.6}$$

We must find, in each case, subdomains of the arbitrary parameter space for which the two conditions on $\gamma_1\gamma_2$ and on the shock limits are satisfied. In Appendix B we determine two classes of positive densities corresponding to the two cases $\gamma_1\gamma_2 \lessgtr 0$ (Theorems 1 and 2). Here, for simplicity, we briefly report the result for the case $\gamma_1\gamma_2<0, \Sigma_{ij}>0$. As mentioned above, we choose $1<\xi_1<2$, $\xi_2=-\xi_1$ and, applying the results of Section 2, the Σ_{ij} are positive if n_{0i}, ξ_1, and $-\xi_1$ satisfy the constraints written down in (A.8a):

$$\begin{aligned} &\text{if}\quad \bar{n}_{12}=(2-\xi_1)/(2+\xi_1)<n_{02}/n_{01}<1/\bar{n}_{12} \\ &\text{and if}\quad n_{03}/n_{01}>\bar{n}_{13}=(2-\xi_1)/(1-\xi_1)\rightarrow \Sigma_{ij}>0 \end{aligned} \tag{3.7}$$

In Lemma 7 of Appendix B we find two possible conditions for $\gamma_1\gamma_2<0$. Let us define

$$X=(n_{02}-n_{01}\bar{n}_{12})/(n_{01}-n_{02}\bar{n}_{12}), \qquad Z=3+4(1-\xi^2)\,n_{01}(4-\xi^2)$$

Then $\gamma_1\gamma_2<0$ in the two cases

$$\text{(i)}\quad 1<n_{02}/n_{01}<1/\bar{n}_{12},\quad Z<-X\quad\text{or}\quad -1/X<Z<0 \tag{3.8a}$$

$$\text{(ii)}\quad \bar{n}_{12}<n_{02}/n_{01}<1,\quad Z<-1/X\quad\text{or}\quad -X<Z<0 \tag{3.8b}$$

If the arbitrary parameters satisfy either (3.7), (3.8a) or (3.7), (3.8b), then $\gamma_1\gamma_2<0$, $\Sigma_{ij}>0$, and the densities N_i are positive.

3.3. Macroscopic Quantities

For the jth similarity components we define the mass $\mathcal{M}_j=m_0+m_j/D_j$, momentum $\mathcal{J}_j=j_0+j_j/D_j$, energy $\mathcal{E}_j=\tilde{e}_0+\tilde{e}_j/D_j$, temperature $\mathcal{T}e_j=2\mathcal{E}_j/\mathcal{M}_j-(\mathcal{J}_j/\mathcal{M}_j)^2$, and the associated limits when $\eta_j\rightarrow-\infty$ or $D_j\rightarrow 1$: $M_j=m_0+m_j$, $J_j=j_0+j_j$, $E_j=\tilde{e}_0+\tilde{e}_j$, $Te_j=2E_j/M_j-(J_j/M_j)^2$. For the $(1+1)$-dimensional solutions we define the mass $\mathcal{M}=m_0+\sum m_j/D_j$, momentum $\mathcal{J}=j_0+\sum j_j/D_j$, energy $\mathcal{E}=\tilde{e}_0+\sum\tilde{e}_j/D_j$, and temperature $\mathcal{T}e=2\mathcal{E}/\mathcal{M}-(\mathcal{J}/\mathcal{M})^2$. For the $\xi_1+\xi_2=0$, $\gamma_1\gamma_2<0$ classes of solutions, the two limits when $|x|\rightarrow\infty$ are just the previous M_j and Te_j.

3.4. Numerical Calculations

In Fig. 2a, for $\xi_1=1.9$, $n_{01}=1$, $n_{02}=23.4$, $n_{03}=9.1$, we present the two similarity shock wave components with a variable η which is either $x-\xi_1 t$ for the first component, $x-\xi_2 t=x+\xi_1 t$ for the second one, or x at $t=0$ for both. We normalize mass and temperature by their ratios to the highest values $M_1=m_0+m_1$ and Te_1. We see that the two shocks are compressive, traveling in opposite direction (arrays), and we observe an overshoot of the temperature for the first component. Figure 2b represents the $(1+1)$-dimensional collision or superposition of the two previous shock waves.

We still normalize mass and temperature by the highest asymptotic values M_1, Te_1. We observe a bump for the temperature at initial time or at small time values. Due to $\gamma_1\gamma_2<0$, $\xi_1\xi_2=-\xi_1^2<0$ and the present choice $\gamma_j\xi_j>0$, the two exponentials $(\exp -\gamma_j\eta_j t)$ decrease when t increases, $D_j\to 1$, and the equilibrium Maxwellian becomes $m_0+m_1+m_2=M_0$ for

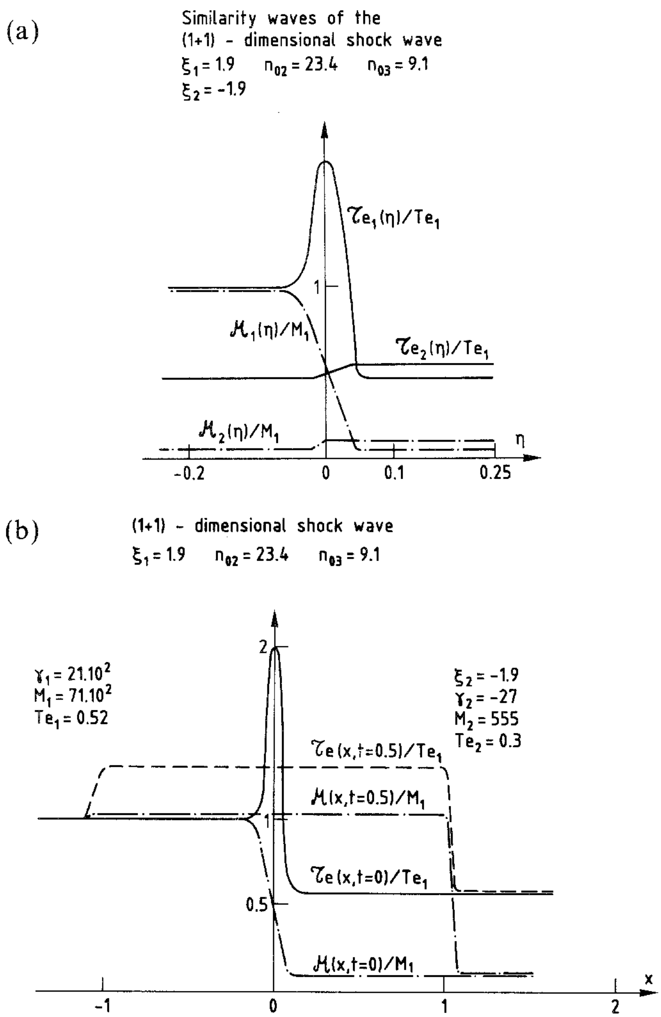


Fig. 2. Results for $(1+1)$-dimensional shock waves. (a) The two similarity components with $\xi=\pm 1.9$ correspond to compressive shocks and we see a temperature overshoot for one component. (b) Collision or superposition of the two components. The temperature overshoot present at $t=0$ decreases and spreads out when t is growing. The Maxwellian equilibrium state appears and we see the moving of the shock.

the mass and $2(e_0+\sum e_i)/M_0-(j_0+\sum j_i)^2/M_0^2$ for the temperature. Contrary to the similarity shock waves, the $(1+1)$-dimensional profiles change with the time. The bump decreases and for sufficiently large time the Maxwellian equilibrium state appears and spreads out. We also observe the moving of the shock.

APPENDIX A. EXPONENTIAL-TYPE SIMILARITY SHOCK WAVES

We seek exponential-type similarity waves with the variable $\eta=x-\xi t$ bounded on the η axis and recall a previous proof[4,14]. From the linear equations (1.1) we see that all $\partial_\eta N_i$ are proportional. We can write $N_i=n_{0i}+n_i/D(\eta)$ and get a Ricatti equation:

$$aD_\eta+a_0+a_1D+a_2D^2=0$$

$$a_0=n_2n_3-n_1n_4$$

$$a_1=n_{02}n_3+n_{03}n_2-n_{01}n_4-n_{04}n_1$$

$$a_2=n_{02}n_{03}-n_{01}n_{04}$$

$$a=n_1(2-\xi)=-n_4(1+\xi)=n_2(2+\xi)=n_3(\xi-1)$$

(i) If $a_2=0$, the solution is a constant plus an exponential $D=-a_0/a_1+d\exp(-a_1\eta/a)$; (ii) if $a_2\neq 0$, putting $D=(a/a_2)\,\partial_\eta\log E$, then E is a sum of two exponentials, $\exp(\lambda_i\eta)$. If $\lambda_1\neq\lambda_2$, we find $D^{-1}=c_1+c_2/[1+\exp(\lambda_1-\lambda_2)\,\eta]$. If $\lambda_1=\lambda_2=-a_1/2a$, the two independent solutions are $\exp\lambda\eta$, $\eta\exp\lambda\eta$, leading to power-type solutions for D, which are excluded.

We determine the similarity solutions

$$N_i=n_{0i}+n_i/D, \qquad D=1+de^{\gamma\eta}, \qquad \eta=x-\xi t, \qquad d=1 \tag{A.1}$$

of the nonlinear system (1.1) with $l_i=\partial_t+a_i\,\partial_x$ and study the properties of the macroscopic quantities: mass $\mathcal{M}$, momentum $\mathcal{J}$, velocity $\mathcal{U}=\mathcal{J}/\mathcal{M}$, shock velocity $\mathcal{V}=\mathcal{U}-\xi$, sound waves $\mathcal{W}=\mathcal{U}-\xi_{Ma}$, temperature $\mathcal{T}e=2\mathcal{E}/\mathcal{M}-\mathcal{U}^2$, local entropy $\mathcal{H}=\sum N_i\log N_i$, and shock entropy $\mathcal{H}_1=\sum(-\xi+a_i)\,N_i\log N_i$.

A.1. Algebraic Determination

The 10 parameters n_{0i}, n_2, γ, ξ satisfy six independent relations,

$$\gamma n_1(2-\xi)=-\gamma n_4(1+\xi)=n_2\gamma(2+\xi)=n_3\gamma(\xi-1)=n_2n_3-n_1n_4$$
$$=n_{01}n_4+n_{04}n_1-n_{02}n_3-n_{03}n_2 \tag{A.2}$$

and the relation for the Maxwellians Ma_0, Ma_s defined by the n_{0i} and $s_{0i} = n_{0i} + n_i$

$$n_{02} n_{03} = n_{01} n_{04}, \qquad s_{02} s_{03} = s_{01} s_{04} \tag{A.3}$$

We choose for the four arbitrary parameters

$$\xi, \quad n_{0i} > 0, \quad i = 1, 2, 3 \rightarrow n_{04} > 0 \tag{A.4}$$

We introduce intermediate ξ-dependent parameters deduced from (A.2),

$$\begin{aligned} \bar{n}_2 &= n_2/n_1 = (2-\xi)/(2+\xi) \\ \bar{n}_3 &= n_3/n_1 = (2-\xi)/(\xi-1) \\ \bar{n}_4 &= n_4/n_1 = (\xi-2)/(\xi+1) \\ \bar{\gamma} &= \gamma/n_1 = 2\xi/(2+\xi)(\xi^2-1) \end{aligned} \tag{A.5}$$

obtain n_1 as a function of the four arbitrary parameters,

$$n_1(2-\xi)\,\bar{\gamma} = n_{01}\bar{n}_4 + n_{04} - n_{02}\bar{n}_3 - n_{03}\bar{n}_2 \tag{A.6}$$

and reconstruct the original parameters $n_i = \bar{n}_i n_1$, $\gamma = \bar{\gamma} n_1$.

A.2. Invariance Properties with the Transforms $\mathscr{T}_1$, $\mathscr{T}_2$

(i) $\mathscr{T}_1: \xi \rightarrow -\xi$, $\gamma \rightarrow -\gamma$, $n_i \rightarrow n_{i+1}$, $n_{0i} \rightarrow n_{0i+1}$, $i = 1$ and 3. The relations (A.2), (A.3) are invariant by $\mathscr{T}_1$ and *consequently we can study the* $\xi > 0$ *solutions and deduce the* $\xi < 0$ *ones.* We note

$$\begin{aligned} \mathscr{T}_1 N_i(x,t) &= N_{i+1}(-x,t), \qquad i = 1, 3 \\ \mathscr{T}_1 \mathscr{M}(x,t) &= \mathscr{M}(-x,t) \\ \mathscr{T}_1 \mathscr{J}(x,t) &= -\mathscr{J}(-x,t) \\ \mathscr{T}_1 \mathscr{E} &= \mathscr{E}(-x,t) \\ \mathscr{T}_1 \mathscr{T}e(x,t) &= \mathscr{T}e(-x,t) \end{aligned}$$

(ii) $\mathscr{T}_2: n_{0i} \rightarrow s_{0i}$, $n_i \rightarrow s_i = -n_i$, $\gamma \rightarrow -\gamma$, $\xi \rightarrow \xi$. Then $\mathscr{T}_2 N_i = n_{0i} + n_i - n_i/(1 + e^{-\gamma\eta}) = N_i$, $\mathscr{T}_2 \mathscr{M} = \mathscr{M}$, and the densities are invariant. We define for the Maxwellians

$$\begin{aligned} M_0 &= n_{01} + n_{02} + 2(n_{03} + n_{04}) \\ M_s &= s_{01} + s_{02} + 2(s_{03} + s_{04}) \\ M &= n_1 + n_2 + 2(n_3 + n_4) \end{aligned}$$

and get $\mathscr{T}_2 M_0 = M_s = M_0 + M$. For the solutions $\mathscr{M} = M_0 + M/D$ with $M > 0$, applying $\mathscr{T}_2$, we deduce those with $M < 0$, interchanging the Maxwellians.

A.3. Study of $N_i > 0$ or $n_{0i} > s_{0i} > 0$

We have from (A.4) $n_{0i} > 0$ and for $s_{0i} = n_{0i} + n_1 \bar{n}_i$ we apply (A.5), (A.6),

$$\begin{aligned} s_{01} &= A(n_{02}/n_{01} - \bar{n}_2)(n_{03} - \bar{n}_3 n_{01}) \\ s_{02} &= \bar{n}_2 A(n_{02}/n_{01} - \bar{n}_2)(n_{03} - n_{01}\bar{n}_4/\bar{n}_2) \\ s_{03} &= \bar{n}_3 A(n_{02}/n_{01} - \bar{n}_4/\bar{n}_3)(n_{03} - \bar{n}_3 n_{01}) \\ A &= (\xi^2 - 1)/2\xi\bar{n}_2 \end{aligned} \tag{A.7}$$

and recall that $s_{04} = s_{03}s_{02}/s_{01}$. For $|\xi| > 2$ we get $\bar{n}_2 < 0$, $\bar{n}_3 < 0$, $A < 0$, $s_{01} < 0$, whence only $|\xi| < 2$ can lead to $N_i > 0$. Starting with $1 < \xi < 2$ and $0 < \xi < 1$ we respectively find $\bar{n}_2 > 0$, $\bar{n}_3 > 0$, $A > 0$, $\bar{n}_4 < 0$ and $\bar{n}_2 > 0$, $\bar{n}_3 < 0$, $A < 0$, $\bar{n}_4 < 0$. For the $N_i > 0$ we get two classes of $\xi > 0$ solutions and with $\mathscr{T}_1$ deduce the $\xi < 0$ ones:

$$\begin{array}{lll} \text{Class I} & 1 < \xi < 2, & n_{02}/n_{01} > \bar{n}_2, \quad n_{03}/n_{01} > \bar{n}_3 \\ \text{Class II} & -2 < \xi < -1, & n_{02}/n_{01} < \bar{n}_2, \quad n_{03}/n_{01} > \bar{n}_4/\bar{n}_2 \\ \text{Class III} & 0 < \xi < 1, & \bar{n}_4/\bar{n}_3 < n_{02}/n_{01} < \bar{n}_2 < 1 \\ \text{Class IV} & -1 < \xi < 0, & 1 < \bar{n}_2 < n_{02}/n_{01} < \bar{n}_4/\bar{n}_3 \end{array} \tag{A.8a}$$

For instance, applying $\mathscr{T}_1$ to class I, we find

$$\begin{aligned} &1 < -\xi < 2 \\ &n_{01}/n_{02} > (2+\xi)/(2-\xi) = 1/\bar{n}_2 \\ &n_{04}/n_{02} = n_{03}/n_{01} > -(2+\xi)/(\xi+1) = \bar{n}_4/\bar{n}_2 \end{aligned}$$

which defines class II. For a given Ma_0 we can rewrite the ξ intervals (A.8a) leading to $N_i > 0$. Either $n_{02}/n_{01} > 1$ or < 1, which excludes either class III or IV, and only three different ξ intervals can exist. We define a, b and find

$$\begin{aligned} a(n_{0i}) &= (n_{03} + 2n_{01})/(n_{03} + n_{01}) > 1 \\ b(n_{0i}) &= (n_{01} - n_{02})/(n_{01} + n_{02}) \end{aligned}$$

$$\begin{array}{ll} \text{Class I} & \sup(a(n_{0i}), 2b(n_{0i})) \leqslant \xi < 2 \\ \text{Class II} & -2 < \xi \leqslant \inf(-a, 2b) \\ \text{Class III} & n_{01} \geqslant n_{02}, \quad 0 < b \leqslant \xi \leqslant \inf(1, 2b) \\ \text{Class IV} & n_{02} \geqslant n_{01}, \quad \sup(-1, 2b) \leqslant \xi \leqslant b < 0 \end{array} \tag{A.8b}$$

For a given Ma_s, applying $\mathscr{T}_2$ to (A.8b), we obtain, for the same ξ, similar intervals with s_{0i} instead of n_{0i}:

$$\begin{array}{ll} \text{Class I} & \sup\{a(s_{0i}), 2b(s_{0i})\} \leqslant \xi < 2 \\ \text{Class II} & -2 < \xi \leqslant \inf(-a, 2b)... \end{array} \tag{A.8c}$$

From n_1 we define a ξ cubic polynomial $\hat{n}_1(\xi, n_{01},..., n_{04})$ with n_{0i}-dependent coefficients,

$$\hat{n}_1(\xi, n_{0i}) = n_1 2\xi(\xi - 2) = (2 + \xi)(\xi^2 - 1)(n_{04} + n_{01}\bar{n}_4 - n_{02}\bar{n}_3 - n_{03}\bar{n}_2)$$

$$\hat{n}_1(\xi \to -\xi, n_{0i}) = -\hat{n}_1(\xi, n_{0i} \to n_{0i+1}), \qquad i = 1 \text{ and } 3 \tag{A.9}$$

and determine for the ξ limits of the intervals of (A.8b) the corresponding $\hat{n}_1(\xi)$ signs,

$$\hat{n}_1(-2) = -12n_{03} < 0 \to \hat{n}_1(2) > 0$$

$$\hat{n}_1(1) = -6n_{02} < 0 \to \hat{n}_1(-1) > 0$$

$$\hat{n}_1(a(n_{0i})) = 2n_{01}n_{03}a/(n_{03} + n_{01}) > 0 \to \hat{n}_1(-a) < 0$$

$$\hat{n}_1(2b) = 16n_{01}n_{02}b/(n_{01} + n_{02}) \gtrless 0 \qquad \text{if} \quad n_{01} \gtrless n_{02}$$

$$\hat{n}_1(-2b) \lesseqgtr 0 \qquad \text{if} \quad n_{01} \lesseqgtr n_{02}$$

$$\hat{n}_1(b) = -4n_{03}n_{02}b/(n_{01} + n_{02}) \lesseqgtr 0$$

$$\text{if} \quad n_{01} \gtrless n_{02} \to \hat{n}_1(-b) \lesseqgtr 0 \quad \text{if} \quad n_{01} \lesseqgtr n_{02} \tag{A.8d}$$

Similarly, for the s_{0i} parameters let us define a ξ cubic polynomial with $s_{0i} = n_{0i} + n_1\bar{n}_i$ coefficients and which satisfies an important relation with $\hat{n}_1$,

$$\hat{s}_1(\xi, s_{0i}) = (2 + \xi)(\xi^2 - 1)(s_{04} + s_{01}\bar{n}_4 - s_{02}\bar{n}_3 - s_{03}\bar{n}_2) = -\hat{n}_1(\xi, n_{0i}) \tag{A.10}$$

However, applying the $\mathscr{T}_2$ transform to (A.8d), we find the same signs for the ξ limits of the (A.8c) intervals,

$$\hat{s}_1(\xi = -2) = -12s_{03} < 0$$

$$\hat{s}_1(2) > 0, \qquad \hat{s}_1(1) < 0, \qquad \hat{s}_1(-1) > 0, \qquad \hat{s}_1(a(s_{0i})) > 0 \tag{A.8e}$$

$$\hat{s}_1(-a) < 0, \qquad \hat{s}_1(b) \lesseqgtr 0 \quad \text{if} \quad s_{01} \gtrless s_{02},...$$

A.4. Weak Shocks Associated to Ma_0 and Ma_s

Let us define ξ_0, the characteristic velocities associated to Ma_0, corresponding to $n_i = 0$ or $\hat{n}_1(\xi_0) = 0$. From (A.9) three ξ_0 roots exist. From the

$\hat{n}_1(\xi)$ signs given by (A.8d) it follows that one ξ_0 root is in the (A.8b) interval defined for class I for ξ, another in the interval defined by class II for ξ, while the last ξ_0 root is either in the ξ class III or the ξ class IV, depending upon whether $n_{01} \gtrless n_{02}$.

Another way to find the characteristic values ξ_0, ξ_s is to apply the Lax–Whitham theory[11] (see Gratignol[1,11] for the $6\mathbf{V}_i$ model). We linearize (1.1) around M_a with $n_{0i}(1+Z_i(x,t))$ and keep terms linear in Z_i:

$$n_{01} l_1 Z_1 = n_{04} l_4 Z_4 = -n_{02} l_2 Z_2 = -n_{03} l_3 Z_3 = n_{02} n_{03}(Z_2 + Z_3 - Z_1 - Z_4)$$

We obtain $\Delta Z = 0$ with Δ a matrix differential operator and Z a column vector with elements $Z_1,\ldots, Z_4$. The determinant of Δ is the sum of a fourth-order differential operator $(\partial^2_{t^2} - 4\partial^2_{x^2})(\partial^2_{t^2} - \partial^2_{x^2})$ plus a third-order one

$$(\partial^2_{t^2} - 4\partial^2_{x^2})[n_{01}(\partial_t + \partial_x) + n_{02}(\partial_t - \partial_x)] \\ + (\partial^2_{t^2} - \partial^2_{x^2})[n_{03}(\partial_t + 2\partial_x) + n_{04}(\partial_t - 2\partial_x)]$$

Applying these operators to a $(x-\xi t)$-dependent function, we find the roots $\xi = \pm 1, \pm 2$ for the fourth-order operator and the cubic ξ polynomial $\hat{n}_1$ [see (A.9)] for the third-order one. The three $\hat{n}_1 = 0$ roots belong to the three intervals $(-2, -1)$, $(-1, 1)$, $(1, 2)$ with end points given by the roots of the fourth-order operator.

For the characteristic value ξ_s of the other Maxwellian Ma_s with $\hat{s}_1(\xi_1) = 0$ we deduce from (A.8c), (A.8e), and (A.10) that one root is in class III or IV, depending upon whether $s_{01} \gtrless s_{02}$.

A.5. Velocity and Shock Velocity

For the mass $\mathscr{M} = M_0 + M/D$, momentum $\mathscr{J} = J_0 + J/D$, velocity $\mathscr{U} = \mathscr{J}/\mathscr{M}$, and shock velocity $\mathscr{V} = \mathscr{U} - \xi$ we associate for Ma_0

$$M_0 = (n_{01} + n_{02})(1 + 2n_{03}/n_{01}) > 0$$

$$J_0 = 2(n_{01} - n_{02})(1 + n_{03}/n_{01})$$

$$U_0 = J_0/M_0$$

$$M_0 V_0 = J_0 - M_0 \xi$$

$$= 2[n_{01}(1-\xi) - n_{02}(1+\xi)](n_{03}/n_{01} - \tilde{n}_3)$$

$$2\tilde{n}_3[n_{01}(1-\xi) - n_{02}(1+\xi)] = (2+\xi)\, n_{0i} - (2-\xi)\, n_{01}$$

$$\mathscr{T}_1 V_0 = -V_0, \qquad \mathscr{T}_1 \tilde{n}_3 = \tilde{n}_3 \tag{A.11}$$

For class I with $n_{02} > \bar{n}_2$ we find $V_0 < 0$, while for class III we distinguish between $n_{03} \gtrless n_{01}\tilde{n}_3$. For the other classes we apply $\mathscr{T}_1$:

$$\text{Class I} \quad V_0 < 0, \qquad \text{Class II} \quad V_0 > 0, \qquad \text{Class III} \quad V_0 \gtrless 0$$
$$\text{if} \quad n_{03} \gtrless \tilde{n}_3 n_{01} \text{ and the converse for Class IV} \tag{A.8f}$$

For Ma_s we associate $M_s = M_0 + M > 0$, $J_s = J_0 + J$, $U_s = M_s/J_s$, $V_s = U_s - \xi$ linked to Ma_0 with the mass conservation law

$$M_s V_s = M_0 V_0 \qquad \text{or} \qquad J = \xi M \rightarrow V_0 V_s > 0 \tag{A.12}$$

so that V_s has the V_0 sign provided by (A.8f).

From $D = 1 + e^{\gamma\eta}$, the determination when $|\eta| \rightarrow \infty$ of the upstream and downstream needs knowledge of the γ sign,

$$\begin{aligned} \xi M/6 &= \gamma = \hat{n}_1/(\xi^2 - 4)(\xi^2 - 1) \\ \xi(2-\xi)\, M/6 &= (n_{03}/n_{01})(n_{02} - \bar{n}_2 n_{01}) + \bar{n}_4 \bar{n}_3 n_{02} \end{aligned} \tag{A.13}$$

Due to $M\xi\gamma > 0$ we get, for ξ fixed, two possibilities $M \gtrless 0$ called A for $M > 0$ and B for $M < 0$. From the two possible ξ classes, the two V_0 signs for classes III and IV and the $M \gtrless 0$ signs we should have 12 subclasses. However, $\mathscr{T}_1$, $\mathscr{T}_2$ with, respectively, $\xi \leftrightarrow -\xi$ and $M_0 \rightarrow M_s$ or $M > 0 \leftrightarrow M < 0$ allow us to study only three subclasses.

Class IA: $1 < \xi < 2$, $M > 0 \gamma > 0$, $V_0 < 0$, up $\eta = \infty$ Ma_0, down $\eta = -\infty$ Ma_s, $M_0 < M_s$ compressive shock.

Class IIIA: $0 < \xi < 1$, $M > 0$, $\gamma > 0$, $\eta = \infty$ Ma_0, $\eta = -\infty$ Ma_s; $V_0 < 0$ if $n_{03} > \tilde{n}_3 n_{01}$, up Ma_0, down Ma_s, compressive shock; $V_0 > 0$ if $n_{03} < \tilde{n}_3 n_{01}$, up Ma_s, down Ma_0, rarefactive shock.

Classes I and III:

$$M > 0 \qquad \text{if} \quad n_{03}/n_{01} > (\bar{n}_3 n_{02} + \bar{n}_4 n_{01})/(n_{02} - \bar{n}_2 n_{01}) \tag{A.8g}$$

A.6. Sound Velocity

For Ma_0, Ma_s we define the sound velocities $W_0 = U_0 - \xi_0 = V_0 + \xi - \xi_0$, $W_s = U_s - \xi_s$ and, comparing with V_0, V_s, we verify that the supersonic and subsonic flow inequalities are satisfied.

Lemma 1. Class IA: up $|W_0| < |V_0|$, down $|V_s| < |W_s|$, $\xi_0 < \xi < \xi_s$.

We recall: $1 < \xi < 2$, $M > 0 \rightarrow \hat{n}_1(\xi) < 0$ from (A.13) $\rightarrow \hat{s}_{01}(\xi) > 0$ from (A.10), $\hat{n}_1(2) < 0$, $\hat{n}_1(1) > 0$, $\hat{s}_{01}(2) < 0$, $\hat{s}_{01}(1) > 0$ from (A.8d) (A.8c). Further, ξ, ξ_0 and ξ, ξ_s belong to the same intervals (A.8c) with only one

$\hat{n}_1(\xi_0) = 0 = \hat{s}_{01}(\xi_s)$ root. Consequently, $\xi_0 < \xi < \xi_s$ and from $V_s < 0$, $W_s = V_s + \xi - \xi_s < V_s < 0$ we get the subsonic inequality $|W_s| > |V_s|$. If $n_{01} > n_{02}$, then $2b(n_{0i}) < 0 < \xi_0$ and if $n_{01} < n_{02}, \hat{n}_1(2b(n_{0i}) = \xi) > 0$ from (A.8a) and still $2b < \xi$ or $n_{02} > \bar{n}_2(\xi = \xi_0)$. We use this result in $W_0 M_0 = J_0 - \xi_0 M_0$

$$J_0 - \xi_0 M_0 = n_{01}(2 - \xi_0) - n_{02}(2 + \xi_0) + 2(n_{03}/n_{01})[n_{01}(1 - \xi_0) - n_{02}(1 + \xi_0)] \tag{A.14}$$

We get $W_0 < 0$ and from $V_0 < 0$, $W_0 - V_0 = \xi - \xi_0 > 0$ the supersonic inequality $|W_0| < |V_0|$.

Lemma 2. Class IIIB and $V_0 > 0$: up $V_0 > W_0 > 0$, down $W_s > V_s > 0$, $\xi_s < \xi < \xi_0$.

We recall $M < 0 \to \hat{n}_1(\xi) < 0$ (A.13) $\to \hat{s}_{01}(\xi) > 0$ $n_{01} > n_{02}$ (A.8b) $\to b(n_{0i}) > 0 \to \hat{n}_1(b) > 0$ $\hat{n}_1(1) > 0$, $\hat{s}_1(1) > 0$ (A.8d), (A.8e), ξ, ξ_0 and ξ, ξ_s belong to the same intervals with one ξ_0 and ξ_s root. Consequently, $\xi_s < \xi < \xi_0$ and $W_s = V_s + \xi - \xi_s > V_s > 0$ is the subsonic inequality. Further, from (A.8d), $\hat{n}_1(b(n_{0i}) < 0$, $n_{02} < n_{01} \to b(n_{0i}) > \xi_0$, or $n_{02} < n_{01}(1 - \xi_0)/(1 + \xi_0)$. Substituting into (A.14), we get $W_0 > 0$, $V_0 = \xi_0 - \xi + W_0 > W_0$ or the supersonic inequality.

Lemma 3. Class IIIB and $V_0 < 0$: up $-W_0 > -V_0$, down $-V_s > -W_s$.

From $M < 0$ and Lemma 2 we still have $\xi_1 < \xi < \xi_0$ and deduce $-W_0 = -V_0 + \xi - \xi_0 > -V_0$ for the subsonic inequality. From $\hat{n}_1(\xi) < 0$ and (A.9) we find $n_1(\xi) > 0$ and

$$s_{02} - s_{01} - n_{02} + n_{01} = n_1(\bar{n}_2 - 1) = -2\xi n_1 n_{01}/(2 + \varepsilon) < 0$$

Consequently, $s_{02} < s_{01}$, $\hat{s}_1(b(s_{0i})) < 0$ from (A.8e) or $b(s_{0i}) > \xi_s$ or $s_{02} < s_{01}(1 - \xi_1)/(1 + \xi_s)$. We write down an expression similar to (A.14) for $W_s M_s$ and substitute the inequality

$$W_s M_s = s_{01}(2 - \xi_1) - s_{02}(2 + \xi_s) + 2(s_{03}/s_{01})[s_{01}(1 - \xi_s) - s_{02}(1 + \xi_s)] < 0$$

Finally, $-V_s = -W_s + \xi - \xi_s > -W_s$, which is the supersonic inequality.

A.7. Energy and Temperature

We introduce the energy $\mathscr{E} = 2(N_1 + N_2) + N_3 + N_4$ and apply the momentum conservation $J_t + 2\mathscr{E}_x = 0$,

$$\mathscr{E} = E_0 + E/D(\eta), \qquad E_0 = (n_{01} + n_{02})(1 + 2n_{03}/n_{01})$$
$$\partial_\eta(-\xi\mathscr{J} + 2\mathscr{E}) = 0 \to 2E = \xi J \to 2EM = J^2 \tag{A.15}$$

Then, taking into account the last relation (A.15) in the temperature $\mathscr{T}e = 2\mathscr{E}/\mathscr{M} - (\mathscr{J}/\mathscr{M})^2$, the term D^{-2} in the numerator disappears:

$$\mathscr{T}e = (\mathscr{N}_0 + \mathscr{N}/D)/(M_0 + M/D)^2, \qquad \mathscr{N} = 2(E_0 M + EM_0 - JJ_0) = 2MC$$
$$\mathscr{N}_0 = 2E_0 M_0 - J_0^2 = 2(n_{01} + n_{02})^2\, n_{03}/n_{01} + 16 n_{01} n_{02}(1 + n_{03}/n_{01})^2 > 0$$
$$2C = 2E_0 + \xi^2 M_0 - 2\xi\mathscr{J}_0$$
$$= n_{02}[(\xi+2)^2 + 2n_{03}(\xi+1)^2/n_{01}] + n_{01}(\xi-2)^2 + 2n_{03}(\xi-1)^2 > 0 \tag{A.16}$$

To the Maxwellians Ma_0 and Ma_s we associate, when $|\eta| \to \infty$, the temperatures $\mathscr{T}e_0$ and $\mathscr{T}e_s$:

$$\mathscr{T}e_0 = \mathscr{N}_0/M_0^2, \qquad \mathscr{T}e_s = (\mathscr{N}_0 + \mathscr{N})/M_s^2$$
$$(\mathscr{T}e_0 - \mathscr{T}e_s)\, M_0 M_s^2/\mathscr{N}_0 = M[m + 2(1 - 1/\mu)] \tag{A.17}$$
$$\mu = M_0 C/\mathscr{N}_0 > 0, \qquad m = M/M_0$$

and m has the M sign. For compressive or rarefactive shocks the mass increases or decreases across the shock. For class III, $0 < \xi < 1$, we study the property that both $\mathscr{M}$ and $\mathscr{T}e$ are increasing or decreasing together.

Lemma 4. For class III with compressive shock, mass and temperature cannot both increase across the shock.

We choose class IIIA with $M > 0$ or $M_0 < M_s$ and prove that $\mathscr{T}e_0 < \mathscr{T}e_s$ is not possible. For class III, due to $n_{02} < n_{01}$ and (A.11), we have $\mathscr{J}_0 = U_0 M_0 > 0$ and

$$E_0 - \xi^2 M_0/2 = (n_{01} + n_{02})[2 - \xi^2/2 + (1 - \xi^2)\, n_{03}/n_{01}] > 0 \tag{A.18}$$

$\mathscr{T}e_0 < \mathscr{T}e_s$, $M > 0$ lead to $m > 0$, $m < 2(1-\mu)/\mu$, requiring $\mu < 1$

$$(\mu - 1)\, C = (E_0 - \xi^2 M_0/2) - U_0 V_0 M_0 \tag{A.19}$$

From (A.18) the first term is positive and also the second one because from (A.8g), $V_0 < 0$ for compressive shock. It follows that $\mu < 1$ is not possible. For class IIIB with $M < 0$ we use the transform $\mathscr{T}_2$.

Lemma 5. For class III with rarefactive shock, mass and temperature cannot both decrease across the shock for $\xi^2 < 4/5$. For class III, due to $\gamma M > 0$, if $M > 0$, then $\gamma > 0$ and $M_s = \lim_{\eta \to -\infty} \mathscr{M} > M_0 = \lim_{\eta \to \infty} \mathscr{M}$; while if $M < 0$, then $\gamma < 0$ and $M_0 = \lim_{\eta \to -\infty} \mathscr{M} > \mathscr{M}_s = \lim_{\eta \to \infty} \mathscr{M}$. Consequently, for rarefactive shocks $V_0 > 0$ and the last term of (A.19), $-J_0 V_0$, gives a negative contribution. Still assuming $M > 0$, we must have $\mu < 1$ and a negative rhs in (A.19). Applying $(1-\xi)/(1+\xi) < n_{02}/n_{01} < \tilde{n}_2$ and

$$\begin{aligned}
&\xi/2 < (n_{01} - n_{02})/(n_{01} + n_{02}) < \xi \\
&1 < 2(n_{01} + n_{03})/(1 + 2n_{03}) < 2 \to \xi/2 < U_0 < 2\xi \\
&V_0 M_0 = 2(n_{03}/n_{01})[n_{01}(1-\xi) - n_{02}(1+\xi)] + n_{01}(2-\xi) - n_{02}(2+\xi) \\
&\quad \to (\mu - 1)\, C > n_{03}(1-\xi) + n_{04}(1+\xi) \\
&\quad + n_{02}(2 + 4\xi + 3\xi^2/2) + n_{01}(2 - 4\xi + 3\xi^2/2)
\end{aligned} \tag{A.20}$$

with a negative n_{01} term for $\xi > 2/3$. Still using the lower bound for n_{02}/n_{01}, we get that $\mu < 1$ or $\mathscr{T}e_0 < \mathscr{T}e_s$ is not possible for $\xi^2 < 4/5$,

$$(\mu - 1)\, C > 2n_{03}(1-\xi) + n_{01}(4 - 5\xi^2)/(1+\xi) > 0 \qquad \text{for} \quad \xi^2 < 4/5 \tag{A.21}$$

A.8. Overshoot of the Temperature

Neglecting $\mathscr{J}$ in the temperature, then $\mathscr{T}e \to \bar{\mathscr{T}}e = 2\mathscr{E}/\mathscr{M}$ and $\partial_\eta \bar{\mathscr{T}}e = 2\mathscr{M}D^{-2}(D-1)\,\gamma(ME_0 - EM_0)$ has a constant sign. $\bar{\mathscr{T}}e$ is a monotonic ξ-dependent function. Adding $-\mathscr{U}^2$, then $\mathscr{T}e$ can be nonmonotonic. A criterion for an overshoot of $\mathscr{T}e$ is

$$\mathscr{T}e(\eta = 0) = (\mathscr{N}_0 + \mathscr{N}/2)/(M_0 + M/2)^2 > \sup \mathscr{T}e_0, \mathscr{T}e_s \tag{A.22}$$

Restricting our study to the solutions $M_0 \gtrless M_s \to \mathscr{T}e_0 \gtrless \mathscr{T}e_s$, the criterion becomes

$$\begin{aligned}
&\text{Class A} \qquad M > 0, \quad \mathscr{T}e_0 < \mathscr{T}e_s < \mathscr{T}e(0) \\
&\text{Class B} \qquad \mathscr{T}e_s < \mathscr{T}e_0 < \mathscr{T}e(0)
\end{aligned} \tag{A.23}$$

In addition to the sign of $\mathscr{T}e_0 - \mathscr{T}e_s$, provided by (A.17), we find

$$\begin{aligned}
&[\mathscr{T}e(0) - \mathscr{T}e_0]\, M_0(M_0 + M/2)^2/\mathscr{N}_0 \\
&\quad = M(-m/4 - 1 + 1/\mu) \\
&[\mathscr{T}e(0) - \mathscr{T}e_s][M_0 + M)(M_0 + M/2)]^2/\mathscr{N}_0 M_0 \\
&\quad = M[3m/4 + 1 + (m^2/2 - 1)/\mu]
\end{aligned} \tag{A.24}$$

so that the conditions for the overshoot become

$$\begin{aligned}
&\text{Class A} \quad M>0, \quad -3\mu/4+\tfrac{1}{2}[9\mu^2/4-8(\mu-1)]^{1/2} \\
&\qquad\qquad < m < 2(1-\mu)/\mu, \quad \mu<1 \qquad \text{(A.25)} \\
&\text{Class B} \quad M<0, \quad 2< -m\mu/(\mu-1)<4, \quad \mu>1
\end{aligned}$$

A.9. Entropies

We define the shock entropy $\mathscr{H}_1$ and the local entropy $\mathscr{H}$:

$$\mathscr{H}_1=\sum(-\xi+a_i)\,N_i\log N_i \rightarrow \partial_\eta \mathscr{H}_1=(N_2N_3-N_1N_4)\log(N_1N_4/N_2N_3)\leqslant 0$$

$$\mathscr{H}=\sum N_i\log N_i \rightarrow \partial_t\mathscr{H}+\partial_x\sum a_iN_i\log N_i=H_{1\eta}\leqslant 0$$

$\mathscr{H}_1$ is a ξ-monotonic decreasing function, $\mathscr{H}_1(\eta=-\infty)\geqslant\mathscr{H}_1(\eta=\infty)$, while $\mathscr{H}$ is not necessarily monotonic.

APPENDIX B. (1+1)-DIMENSIONAL SHOCK WAVES

We study the $(1+1)$-dimensional solutions which are sums of two similarity waves

$$N_i=n_{0i}+\sum_1^2 n_{ji}/D_j, \qquad D_j=1+d_je^{\gamma_j\eta_j}, \qquad \eta_j=x-\xi_jt, \qquad d_j>0 \quad \text{(B.1)}$$

B.1. Algebraic Determination

The 16 parameters n_{0i}, n_{ji}, γ_j, ξ_j satisfy 12 relations, leaving 4 arbitrary parameters. First we have the (A.3a), (A.3b) similarity wave relations

$$\begin{aligned}
&\gamma_jn_{j1}(2-\xi_j)=-\gamma_jn_{j4}(1+\xi_j)=n_{j2}\gamma_j(2+\xi_j)=n_{j3}\gamma_j(\xi_j-1) \\
&\qquad =n_{j2}n_{j3}-n_{j1}n_{j4}=n_{01}n_{j4}+n_{04}n_{j1}-n_{02}n_{j3}-n_{03}n_{j2} \qquad \text{(B.2a)} \\
&n_{04}=n_{02}n_{03}/n_{01}
\end{aligned}$$

and an additional relation for the sum to be a solution,

$$n_{12}n_{23}+n_{13}n_{22}=n_{11}n_{24}+n_{14}n_{21} \qquad \text{(B.2b)}$$

We still introduce ξ_j-dependent intermediate parameters $\bar{n}_{ji} = n_{ji}/n_{j1}$, $\bar{\gamma}_j/n_{j1}$ and obtain n_{j1}:

$$\bar{n}_{j2} = (2-\xi_j)/(2+\xi_j), \qquad \bar{n}_{j3} = (2-\xi_j)/(\xi_j - 1), \qquad \bar{n}_{j4} = (\xi_j - 2)/(\xi_j + 1)$$
$$2\xi_j = \bar{\gamma}_j(2+\xi_j)(\xi_j^2 - 1), \qquad n_{j1}(2-\xi_j)\,\bar{\gamma}_j = n_{01}\bar{n}_{j4} + n_{04} - n_{02}\bar{n}_{j3} - n_{03}\bar{n}_{j2} \tag{B.3}$$

However, the additional relation $\bar{n}_{12}\bar{n}_{23} + \bar{n}_{13}\bar{n}_{22} = \bar{n}_{24} + \bar{n}_{14}$ shows that the two ξ_j cannot be arbitrary. We choose the four arbitrary parameters

$$\xi_1, \quad n_{0i} > 0, \quad i = 1, 2, 3 \tag{B.4}$$

and determine ξ_2 from ξ_1. Putting $P = \xi_1\xi_2$, $S = \xi_1 + \xi_2$ into the additional relation, we find two possible solutions for ξ_2:

$$S = 0 \quad \text{or} \quad \xi_1 + \xi_2 = 0 \quad \text{and} \quad P^2 + 4 - 13P + 2S^2 = 0 \tag{B.5}$$

We restrict the study to $\xi_2 = -\xi_1$, which means that if ξ_1 is in class I or III of similarity solutions, then ξ_2 is respectively in class II or IV. Then we can construct all nonarbitrary parameters: first n_{j1} and then n_{i0}, γ_j, using (B.3).

B.2. Positive N_i for $\xi_1 + \xi_2 = 0$

If at t finite the N_i limits $|x| \to \infty$ are positive, then[12] with the d_j in D_j we can have $N_i > 0\ \forall x$. Depending on whether $\gamma_1\gamma_2 \gtrless 0$, we have two sets of conditions for the limits

$$\gamma_1\gamma_2 < 0\text{:} \quad \Sigma_{ij} = n_{0i} + n_{ji} > 0$$
$$\gamma_1\gamma_2 > 0\text{:} \quad n_{0i} > 0, \quad \Omega_i = n_{0i} + n_{1i} + n_{2i} > 0 \tag{B.6}$$

For the $\gamma_1\gamma_2$ we have the relation $\gamma_2\gamma_1 = n_{21}/n_{11}\bar{n}_{12}$ or

$$-\gamma_2/\gamma_1 = X^{-1}(X+Z)/(X^{-1}+Z)$$
$$X = (n_{02} - n_{01}\bar{n}_{12})/(n_{01} - n_{02}\bar{n}_{12}) \tag{B.7}$$
$$Z = 3 + 4[(1-\xi^2)/(4-\xi^2)]\, n_{03}/n_{01}$$

B.2.1. Positivity in the Case $\gamma_1\gamma_2 < 0$

Lemma 6. If $1 < \xi_1 < 2$, $\bar{n}_{12} < n_{02}/n_{01} < 1/\bar{n}_{12}$, $n_{03}/n_{01} > \bar{n}_{13}$, then $\Sigma_{ij} > 0$ and $X > 0$. Now ξ_1 and $\xi_2 = -\xi_1$ belong, respectively, to classes I, II of Appendix A and from (A.8a) we have, for $\Sigma_{ij} > 0$, $\bar{n}_{21} < n_{02}/n_{01} <$

$\bar{n}_{22}=1/\bar{n}_{12}$, $n_{03}/n_{01}>\bar{n}_{13}$, and $n_{03}/n_{01}>\bar{n}_{24}/\bar{n}_{22}=\bar{n}_{13}$. In X written down in (B.7), both numerator and denominator are positive.

Lemma 7. If $1<\xi_1<2$, then $\gamma_1\gamma_2<0$ in both cases

$$1<n_{02}/n_{01}<1/\bar{n}_{12}, \qquad Z<-X<0 \quad \text{or} \quad -1/X<Z<0 \tag{B.8a}$$

$$\bar{n}_{12}<n_{02}/n_{01}<1, \qquad Z<-1/X \quad \text{or} \quad -X<Z<0 \tag{B.8b}$$

First, from Lemma 6, $X>0$ in both cases. Second, from (B.7) and $\bar{n}_{12}=(2-\xi_1)/(2+\xi_1)<1$ we see that $X \gtrless 1$ if $n_{02} \gtrless n_{01}$. Third, from (B.7), $\gamma_1\gamma_2<0$ if $Z \notin [-X, -1/X]$.

As an application of Lemmas and 7, we obtain two classes of $N_i>0$ satisfying both $\Sigma_{ij}>0$ and $\gamma_1\gamma_2<0$; for instance, with (B.8a).

Theorem 1a. If $1<\xi_1<2$, $1<n_{02}/n_{01}<(2+\xi_1)/(2-\xi_1)$, $n_{03}/n_{01}>\sup\{(2-\xi_1)/(1-\xi_1), (4-\xi_1^2)(3+X)/4(\xi_1^2-1)\}$, X defined in (B.7), then $N_i>0$.

Another Theorem 1b can be obtained from (B.8b).

B.2.2. Positivity in the Case $\gamma_1\gamma_2>0$. We obtain from the Ω_i of (B.6)

$$\begin{aligned} \Omega_2 &= (n_{03}-12An_{01})/4A \\ A &= (4-\xi_1^2)/16(\xi_1^2-1) \\ \Omega_4 &= 3n_{03}-32An_{01} \\ \Omega_i n_{01} &= \Omega_{i+1}n_{02}, \qquad i=1,3 \end{aligned} \tag{B.9}$$

Lemma 8. $\Omega_i>0$ if $1<\xi_1<2$, and $n_{03}/n_{01}>12A=3(4-\xi^2)/4(\xi^2-1)$.

Recalling that for $1<\xi_1<2$, depending upon whether $1<n_{02}/n_{01}<1/\bar{n}_{12}$ or $\bar{n}_{12}<n_{02}/n_{01}<1$, then either $X>1$ or $0<X<1$, and that $\gamma_1\gamma_2>0$ if Z belongs to the interval $-X$, $-1/X$ we get the following result.

Theorem 2. If $1<\xi_1<2$, $n_{03}/n_{01}>3(4-\xi^2)/4(\xi^2-1)$, and if either

$$1<n_{02}/n_{01}<1/\bar{n}_{12}, \qquad (3+X^{-1})<n_{03}/n_{01}\frac{4(\xi^2-1)}{4-\xi^2}<3+X$$

or

$$\bar{n}_{12}<n_{02}/n_{01}<1, \qquad (3+X)<n_{03}/n_{01}\frac{4(\xi^2-1)}{4-\xi^2}<3+X^{-1}$$

then we have both $\Omega_i>0$, $\gamma_1\gamma_2>0$ and $N_i>0$.

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